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ON INTERACTING CRACKS IN UNIDIRECTIONAL FIBROUS COMPOSITES

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Abstract—A crack interaction method for cracks in anisotropic fibrous composites is presented. The method is based on the idea of superposition of resultant tractions along the crack faces. The unknown tractions are expressed by a series of base functions which describe the fundamental solution of stress induced by a solitary crack in an infinite elastic medium, loaded by any such functions. The method employed herein falls within the scope of two-dimensional elasticity but may analogously be applied to other three-dimensional problems (penny-shaped cracks). From the results of various crack configurations in unidirectional fibrous reinforced materials, it is shown that the stress intensity factors strongly depend on the axial to transverse stiffness ratio. Finally, some typical examples, which highlight specific features of the method in different composite material systems are considered and compared to other existing results.

1. INTRODUCTION

The widespread use of composite materials in modern design applications has resulted in a growing interest in crack problems in anisotropic materials. Various solution techniques have been employed [see for example, the papers by Sih *et al.* (1965), Badaliance and Gupta (1976), Delale and Erdogan (1977), Zang and Gudmundson (1991) and Binienda (1991)]. However, most of these approaches are often complex and limited to rather simple geometries.

The principal idea of the present approach is to extend the work of Benveniste *et al.* (1989) to the realm of crack interaction in fibrous composite materials. The method of solution uses a superposition technique which replaces a configuration of N cracks with N different problems, each of which considers a single crack loaded by unknown traction in an otherwise continuous infinite medium. The first step in the method is to choose a polynomial expansion for the unknown crack-line traction. This polynomial is a weighted sum of other polynomials that are called base functions in the sequel. The accuracy of the polynomial expansion can be controlled by using base functions of suitable orders. The second step is to compute the stress fields due to a single crack in an infinite medium loaded by each of the above base functions individually. For convenience, Legendre polynomials are used to represent the base functions. However, other choices of base functions are possible as shown by Benveniste *et al.* (1989) for the case of an H-crack configuration. Finally, the method uses the above information to reduce the original problem to a system of linear equations.

The first section of this paper introduces the superposition equations in the general case, the second and third sections, respectively, deal with specific examples which have been solved in the literature by different methods.

2. METHOD OF SUPERPOSITION

Consider the general case of an infinite plate of elastic solid with N randomly oriented cracks, subjected to arbitrary external loads p_i^0 ($i = 1, N$). The distribution of p_i^0 is assumed to be symmetric with respect to the crack faces. Let d_i be the half crack length and (x_i, y_i) be the local reference system centered at the middle of the crack. The (x_i, y_i) is chosen such that the x_i-axis coincides with the crack line. The function $p_{ij}(x_j)$ denotes the normal traction induced by crack i along the imaginary location of crack j when crack i is loaded by $P_i(x_i)$, and $s_{ij}(x_j)$ denotes the shear traction induced by crack i along the imaginary location of crack j when crack i is loaded by $S_i(x_i)$. By superposition, the problem of N cracks will be represented by an equivalent N different problems, each of which considers a single crack in an infinite medium, but loaded by unknown normal and shear tractions P_i, S_i ($i = 1, N$) respectively. The terms P_i and S_i can be regarded as the overall tractions (after interaction) generated at each crack face by the applied external load and by tractions induced by mutual interaction with cracks j ($j = 1, N, j \neq i$). The solution of the original problem is then obtained by superposing the individual solutions of each of the N problems separately. The corresponding system of equations, labeled as superposition equations in Benveniste *et al.* (1989), takes the form:

$$p_i^0(x_i) = P_i(x_i) + \sum_{j=1}^{N} p_{ji}(x_i), \quad s_i^0(x_i) = S_i(x_i) + \sum_{j=1}^{N} s_{ji}(x_i), \quad i \neq j. \tag{1}$$

There are several ways in the literature to represent the unknown tractions P_i and S_i. However, for simplicity, the Legendre polynomial which is referred to by base functions in the sequel is used herein:

$$P_i(x_i) = \sum_{n=0}^{N} a_n^{(i)}[-L_n^{(i)}(\xi_i)], \quad S_i(x_i) = \sum_{n=0}^{N} b_n^{(i)}[-L_n^{(i)}(\xi_i)], \tag{2}$$

where $\xi_i = (x_i/d_i)$ is a normalized parameter and $L_n^{(i)}$ are the Legendre polynomials defined by:

$$L_n(\xi_i) = \frac{1}{2^n n!}\frac{d^n}{d\xi_i^n}(\xi_i^2 - 1)^n, \quad n = 0, 1, 2, 3, \ldots \tag{3}$$

Note that the minus sign convention in eqn (2) indicates that each of the base functions $L_n^{(i)}$ is used to promote a crack opening mode. On the other hand, crack closure is not dealt with in the present work.

In general, depending on the orientation of crack j, a normal (or shear) traction applied at the faces of crack i will induce normal and/or shear traction along the line of crack j. Accordingly, a set of influence functions which describe the tractions at the line of crack i, caused by the base functions $-L_n^{(j)}(\xi_j)$ applied at crack j are defined. Four such functions are needed:

- $f_{ji}^{(n)}(x_j)$ is *normal* stress induced at location j by a *normal* stress of order n applied at crack i;
- $g_{ji}^{(n)}(x_j)$ is *normal* stress induced at location j by a *shear* stress of order n applied at crack i;
- $h_{ji}^{(n)}(x_j)$ is *shear* stress induced at location j by a *normal* stress of order n applied at crack i;
- $q_{ji}^{(n)}(x_j)$ is *shear* stress induced at location j by a *shear* stress of order n applied at crack i.

The above influence functions are determined from the solution of a single crack in an infinite medium loaded by concentrated unit loads (see Fig. A1 in the Appendix).

The induced tractions $p_{ji}(x_i)$ and $s_{ji}(x_i)$ can now be written in terms of the influence functions:

$$p_{ji}(x_i) = \sum_{n=0}^{M} [a_n^{(j)} f_{ji}^{(n)}(x_i) + b_n^{(j)} g_{ji}^{(n)}(x_i)],$$

$$s_{ji}(x_i) = \sum_{n=0}^{M} [a_n^{(j)} h_{ji}^{(n)}(x_i) + b_n^{(j)} q_{ji}^{(n)}(x_i)], \tag{4}$$

where $a_n^{(j)}$ and $b_n^{(j)}$ are unknown weight coefficients to be determined. Substituting (4) and (2) into (1) provides

$$p_i^0(x_i) = \sum_{n=0}^{N} a_n^{(i)}[-L_n^{(i)}(\xi_i)] + \sum_{j=1}^{N}\sum_{n=0}^{M} [a_n^{(j)} f_{ji}^{(n)}(x_i) + b_n^{(j)} g_{ji}^{(n)}(x_i)],$$

$$s_i^0(x_i) = \sum_{n=0}^{N} b_n^{(i)}[-L_n^{(i)}(\xi_i)] + \sum_{j=1}^{N}\sum_{n=0}^{M} [a_n^{(j)} h_{ji}^{(n)}(x_i) + b_n^{(j)} q_{ji}^{(n)}(x_i)], \quad \text{with } j \neq i. \tag{5}$$

It is necessary to have an identical number of equations and unknowns $a_n^{(i)}$, $b_n^{(i)}$. Therefore, each of eqns (5) is multiplied by $L_k(\xi_i)$, $k = 0, 1, 2, \ldots, M$ and integrated with respect to x_i in the interval $-d_i$ to $+d_i$. Using the orthogonality condition of Legendre polynomials

$$\int_{-1}^{+1} L_N(\xi)L_K(\xi)\,d\xi = \delta_{NK}[2/(2K+1)], \tag{6}$$

where δ_{NK} is the Kronecker delta, and the non-dimensional parameter $\xi_i = x_i/d_i$, one finds:

$$p_i^{(k)} = -a_k^{(i)}[2d_i/(2K+1)] + \sum_{j=1}^{N}\sum_{n=0}^{M} [a_n^{(j)} F_{ji}^{(n,k)} + b_n^{(j)} G_{ji}^{(n,k)}],$$

$$s_i^{(k)} = -b_k^{(i)}[2d_i/(2K+1)] + \sum_{j=1}^{N}\sum_{n=0}^{M} [a_n^{(j)} H_{ji}^{(n,k)} + b_n^{(j)} Q_{ji}^{(n,k)}], \quad \text{with } j \neq i \tag{7}$$

where the following definitions are used:

$$p_i^{(k)} = \int_{-d_i}^{d_i} p_i^0(x_i) L_K^{(i)}(x_i/d_i)\,dx_i, \quad s_i^{(k)} = \int_{-d_i}^{d_i} s_i^0(x_i) L_K^{(i)}(x_i/d_i)\,dx_i, \tag{8}$$

together with the formula

$$Z_{ji}^{(n,k)} = \int_{-d_i}^{d_i} z_i^{(n)}(x_i) L_K^{(i)}(x_i/d_i)\,dx_i. \tag{9}$$

Where $Z_{ji}^{(n,k)}$ assumes in turn, the values of $F_{ji}^{(n,k)}$, $G_{ji}^{(n,k)}$, $H_{ji}^{(n,k)}$ and $Q_{ji}^{(n,k)}$, while $z_{ji}^{(n,k)}$ assumes in turn, the values of $f_{ji}^{(n)}$, $g_{ji}^{(n)}$, $h_{ji}^{(n)}$ and $q_{ji}^{(n)}$. Recall that $i = 1, 2, \ldots, N$; $j = 1, 2, \ldots, N$; $k = 0, 1, 2, \ldots, M$; and $n = 0, 1, 2, \ldots, M$. Finally, eqns (5) provide $2N(M+1)$ equations for the $N(M+1)$ unknowns $a_k^{(i)}$, and $N(M+1)$ unknowns $b_k^{(i)}$ needed for the evaluation of tractions p_{ji} and s_{ji} in (4). The stress intensity factors can be evaluated from the well-known expression for the point load solution

$$K_{(I,II)}^{\pm} = \frac{1}{\sqrt{\pi d_i}} \int_{-d_i}^{+d_i} \left(\frac{d_i \pm S_i}{d_i \mp S_i}\right)^{1/2} (P(s_i), S(s_i))\,ds_i, \tag{10}$$

where $\pm$ denotes the stress intensity factor at the inner/outer crack tips and $P(s_i)$, $S(s_i)$ are the overall normal and shear traction, respectively, at the faces of crack i given by (1).

3. APPLICATIONS

As a first example, the problem of infinite plate with two separate cracks as shown in Fig. 1 is considered. This problem has been solved by Badaliance and Gupta (1976) for the isotropic case and by Binienda *et al.* (1991) for the corresponding orthotropic case using the singular integrals technique. The purpose of analysing this example herein is to provide verification of the present scheme against other approaches and techniques. The material used in Binienda *et al.* (1991) is a Gr/Epoxy with the following elastic constants:

$$E_A = 21.08 \times 10^6 \text{ psi}, \quad E_T = 5.08 \times 10^6 \text{ psi},$$
$$G_T = 1.5 \times 10^6 \text{ psi}, \quad G_{AT} = 0.98 \times 10^6 \text{ psi}, \quad \nu_{AT} = 0.3, \tag{11}$$

where subscripts A and T denote axial and transverse direction, respectively.

According to Fig. 1, crack 2 is aligned with the fiber direction such that its local reference system (x_2, y_2) coincides with the principal axes of elastic symmetry. In the sequel, the position of crack 2 is referred to as on-axis. On the other hand, crack 1 which remains in the horizontal plane at an angle θ from the fiber axis $(\bar{x}_1 \bar{x}_2 = \theta)$ is referred to as off-axis. Note that the material anisotropy for each crack system is a function of crack orientation with respect to the fiber-axis, and on-axis cracks admit purely imaginary roots while off-axis cracks admit general complex roots (see Appendix).

First, consider the case where the external applied load is a remote uniform tension. By superposition, the far-field uniform tension is resolved on each crack face and stresses vanish at infinity; the equivalent problem is shown in Fig. 1(b). The problem formulation will be adopted from Section 2.

With reference to eqn (1), the resulting superposition equations are schematically described in Figs 1(c)–(e) and are given below as:

Crack 1:

$$-p^{\infty} = P_1(x_1) + p_{21}(x_1), \quad 0 = S_1(x_1) + s_{21}(x_1); \tag{12}$$

Crack 2:

$$-p^{\infty}\cos^2(\theta) = P_2(x_2) + p_{12}(x_2), \quad -p^{\infty}\sin(\theta)\cos(\theta) = S_2(x_2) + s_{12}(x_2); \tag{13}$$

where the unknown tractions $P_i(x_i)$ and $S_i(x_i)$, $(i = 1, 2)$ are expressed by a combination of Legendre polynomials as:

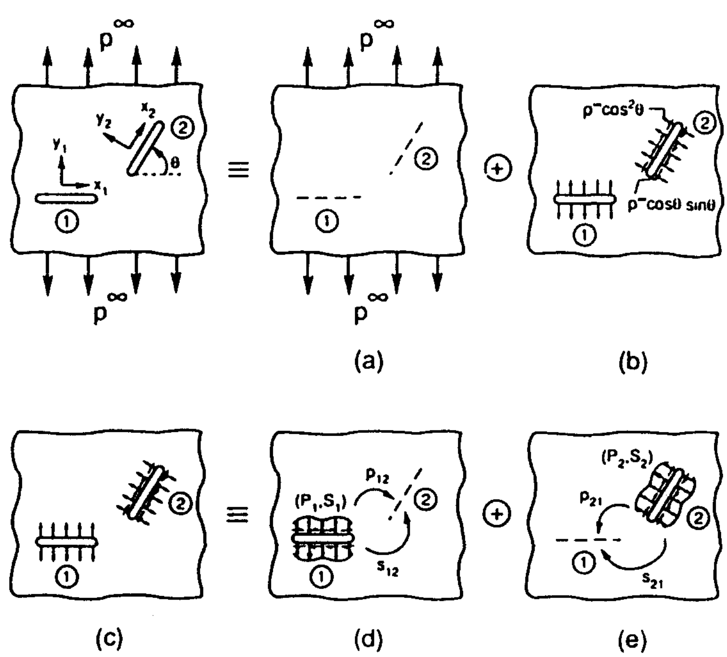

Fig. 1. The two-cracks problem and the schematic of superposition technique.

$$P_i(x_i) = \sum_{n=0}^{1} a_n^{(i)}[-L_n^{(i)}(\xi_i)], \quad S_i(x_i) = \sum_{n=0}^{1} b_n^{(i)}[-L_n^{(i)}(\xi_i)], \quad (i = 1, 2). \tag{14}$$

For simplicity, the first and second order of Legendre polynomials (uniform, linear) are selected here. In actual calculations, one could choose a higher order of L_n as required.

In the second step of the solution, the influence functions [see Zarzour (1990)] are also expanded in terms of L_n. Therefore, with reference to (4), it follows that

$$p_{ij}(x_i) = \sum_{n=0}^{1} [a_n^{(i)} f_{ij}^{(n)}(x_j) + b_n^{(i)} g_{ij}^{(n)}(x_j)], \quad (i = 1, 2; j = 1, 2; i \neq j),$$

$$s_{ij}(x_i) = \sum_{n=0}^{1} [a_n^{(i)} h_{ij}^{(n)}(x_j) + b_n^{(i)} q_{ij}^{(n)}(x_j)]. \tag{15}$$

Substitution of (15) and (14) into (12) and (13) for crack 1 and crack 2, respectively, gives:

Crack 1:

$$-p^{\infty} = \sum_{n=0}^{1} a_n^{(1)}[-L_n^{(1)}(\xi_1)] + \sum_{n=0}^{1} [a_n^{(2)} f_{21}^{(n)}(x_1) + b_n^{(2)} g_{21}^{(n)}(x_1)],$$

$$0 = \sum_{n=0}^{1} b_n^{(1)}[-L_n^{(1)}(\xi_1)] + \sum_{n=0}^{1} [a_n^{(2)} h_{21}^{(n)}(x_1) + b_n^{(2)} q_{21}^{(n)}(x_1)]; \tag{16}$$

Crack 2:

$$-p^{\infty} \cos^2(\theta) = \sum_{n=0}^{1} a_n^{(2)}[-L_n^{(2)}(\xi_2)] + \sum_{n=0}^{1} [a_n^{(1)} f_{12}^{(n)}(x_2) + b_n^{(1)} g_{12}^{(n)}(x_2)],$$

$$-p^{\infty} \sin(\theta)\cos(\theta) = \sum_{n=0}^{1} b_n^{(2)}[-L_n^{(2)}(\xi_2)] + \sum_{n=0}^{1} [b_n^{(1)} h_{12}^{(n)}(x_2) + b_n^{(2)} q_{12}^{(n)}(x_2)]. \tag{17}$$

Multiply (16) and (17) by $L_k^{(i)}$, $k = 0, 1$ and integrate from $-d_i$ to $+d_i$ to provide a system of algebraic equations with certain unknown coefficients $a_n^{(i)}$, $b_n^{(i)}$. The corresponding equations are:

$$p_i^{(k)} = -a_k^{(i)}[2d_i/(2k+1)] + \sum_{n=0}^{1} [a_n^{(j)} F_{ji}^{(n,k)} + b_n^{(j)} G_{ji}^{(n,k)}], \quad (i = 1, 2; j = 1, 2; i \neq j),$$

$$s_i^{(k)} = -b_k^{(i)}[2d_i/(2k+1)] + \sum_{n=0}^{1} [a_n^{(j)} H_{ji}^{(n,k)} + b_n^{(j)} Q_{ji}^{(n,k)}], \tag{18}$$

where $F_{ji}^{(n,k)}$, $G_{ji}^{(n,k)}$, $H_{ji}^{(n,k)}$ and $Q_{ji}^{(n,k)}$ are given by (9).

Finally, the stress intensity factor at the inner/outer tip for each crack can be obtained in the following way:

Crack 1:

$$k_{\mathrm{I}}^{\pm} = (\pi d_1)^{-1/2} \int_{-d_1}^{+d_1} \left(\frac{d_1 \pm s}{d_1 \mp s}\right)^{1/2} \left(p^{\infty} + \sum_{n=0}^{1} (a_n^{(2)} f_{21}^{(n)}(s) + b_n^{(2)} g_{21}^{(n)}(s))\right) \mathrm{d}s,$$

$$k_{\mathrm{II}}^{\pm} = (\pi d_1)^{-1/2} \int_{-d_1}^{+d_1} \left(\frac{d_1 \pm s}{d_1 \mp s}\right)^{1/2} \sum_{n=0}^{1} (a_n^{(2)} h_{21}^{(n)}(s) + b_n^{(2)} q_{21}^{(n)}(s))\, \mathrm{d}s. \tag{19}$$

Crack 2:

$$k_{\mathrm{I}}^{\mp} = (\pi d_2)^{-1/2} \int_{-d_2}^{+d_2} \left(\frac{d_2 \pm s}{d_2 \mp s}\right)^{1/2} \left(p^{\infty} \cos^2(\theta) + \sum_{n=0}^{1} \left(a_n^{(1)} f_{12}^{(n)}(s) + b_n^{(1)} g_{12}^{(n)}(s) \right) \right) \mathrm{d}s,$$

$$k_{\mathrm{II}}^{\mp} = (\pi d_2)^{-1/2} \int_{-d_2}^{+d_2} \left(\frac{d_2 \pm s}{d_2 \mp s}\right)^{1/2} \left(p^{\infty} \sin(\theta)\cos(\theta) + \sum_{n=0}^{1} \left(a_n^{(1)} h_{12}^{(n)}(s) + b_n^{(1)} q_{12}^{(n)}(s) \right) \right) \mathrm{d}s, \qquad (20)$$

where s is a variable of integration.

4. NUMERICAL RESULTS

Hereafter, derivations of the stress intensity factors for the previous example of a two-cracks problem as well as several other examples in both isotropic and orthotropic materials are developed.

4.1. *Two-crack problem under uniform axial tension*

Figure 2 illustrates the crack configuration and the loading conditions, and presents the stress intensity factor results. These are the K_{I} and K_{II} factors at the outer tip A of crack 1 and outer tip B of crack 2 for the isotropic case. Similarly, Fig. 3 presents the same results for the orthotropic material where the elastic constants are described in (11). The stress intensity factors are normalized by $K_{\mathrm{I}}^0 (K_{\mathrm{I}}^0 = p^{\infty}\sqrt{\pi a})$ and given for the full range of $\theta(\theta = 0^{\circ}, \ldots, 90^{\circ})$. As mentioned before, the Legendre polynomial is truncated at $n = 3$ which corresponds to a cubic term. In the isotropic case, Fig. 2, the results are compared to those found by Binienda *et al.* (1991) who used the method of singular integral equations with generalized Cauchy kernels. A close agreement was found for $K_{\mathrm{I}}(A)$, $K_{\mathrm{II}}(A)$ and for $K_{\mathrm{I}}(B)$, $K_{\mathrm{II}}(B)$. In the orthotropic case, Fig. 3, the fiber orientation is aligned with crack 2 at an angle θ from crack 1. The results, which compare well with Binienda *et al.* (1991) indicate that for this configuration, the material influence is rather weak and the stress intensity factors at the tips of both crack 1 and crack 2 are very similar to those of the corresponding isotropic case.

4.2. *An H-crack under uniform axial stress*

Crack branching phenomena such as H-crack or T-crack shapes are often found in fracture problems of unidirectional composites. The stress intensity factors at the tips of these cracks are usually informative about crack stability and fracture toughness

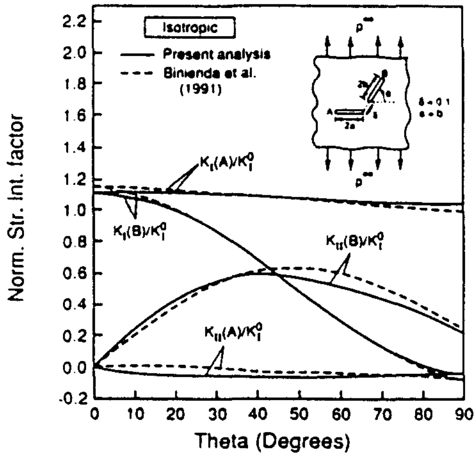

Fig. 2. The stress intensity factors of a two-cracks problem as a function of the kink angle, θ. Isotropic case with $a = b$, $\delta = 0.1$.

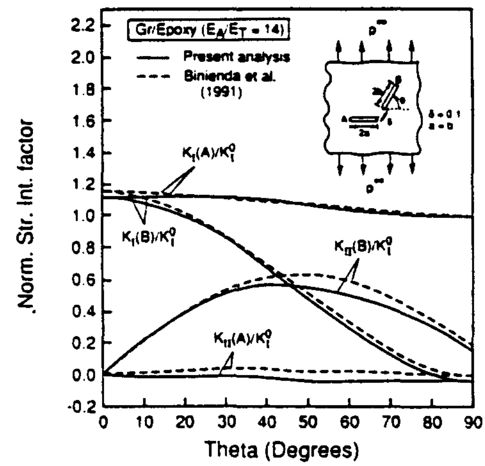

Fig. 3. The stress intensity factors of a two-cracks problem as a function of the kink angle, θ. Orthotropic case with $a = b$, $\delta = 0.1$.

parameters. In a previous study by Dvorak *et al.* (1992) on fracture of metal matrix composites, the analysis required the solution of an H-crack shape in $B/A\ell$ and $FP/A\ell$ composite systems. In the present analysis, the case of an H-crack shape under uniform axial stress is considered. The dominant stress intensity factor $K_{II}(A)$ is normalized by K_I^0 ($K_I^0 = p^{\infty}\sqrt{\pi L}$) and obtained for both the isotropic and orthotropic cases in terms of the ratios R/L as indicated by Fig. 4. As in Benveniste *et al.* (1989), non-linear crack shapes are dealt with by coalescing line cracks into the desired configuration, thus, an H-crack consists of three cracks [see §4 of Benveniste *et al.* (1989)].

In the isotropic case, the present results were previously obtained by Benveniste *et al.* (1989) and compared to other existing solutions in the literature and found to be in close agreement. In the orthotropic case, the fibers are axially oriented along the applied load and two composite systems are considered; Gr/Epoxy and $B/A\ell$. The material properties for $B/A\ell$ are:

$$E_A = 237.3 \times 10^3\ \text{MPa}, \qquad E_T = 143.1 \times 10^3\ \text{MPa},$$

$$G_A = 55 \times 10^3\ \text{MPa}, \qquad \nu_{AT} = 0.21.$$

Shown in Fig. 4 are the four curves obtained for Gr/Epoxy with different ratios of E_A/E_T

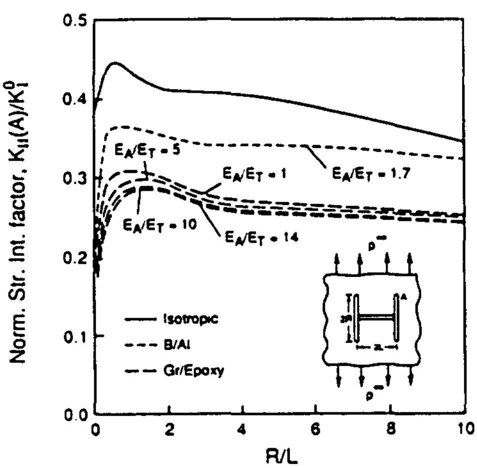

Fig. 4. The H-crack configuration under uniform axial load. The stress intensity factor K_{II} at point A, non-dimensionalized with respect to $K_I^0 = p^{\infty}\sqrt{\pi L}$, plotted versus the ratio R/L. Isotropic and orthotropic results.

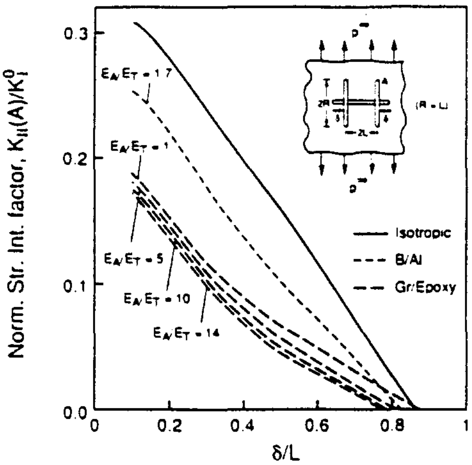

Fig. 5. The extended H-crack configuration under uniform axial load. The stress intensity factor K_{II} at point A, non-dimensionalized with respect to $K_I^0 = p^{\infty}\sqrt{\pi L}$, plotted versus the ratio δ/L. Isotropic and orthotropic results for $R = L$.

while other parameters are kept constant. As the ratio decreases from 14 to 1, the results indicated a slow convergence toward the isotropic curve. On the other hand, the curve representing the B/Al results ($E_A/E_T = 1.7$) showed a closer but not total agreement with the isotropic one.

4.3. *An extended H-crack under uniform axial stress*

This is a special case of the H-crack configuration in which the ratio $R/L = 1$ and the middle crack has propagated by an amount 2δ. Figure 5 shows the results of $K_{II}(A)/K_I^0$ in terms of the ratios δ/L for both the isotropic and orthotropic cases. Similarly to the previous H-crack configuration, the results corresponding to Gr/Epoxy composite show a tendency for a slow convergence toward the isotropic results as the ratios E_A/E_T decrease from 14 to 1 whereas the results corresponding to the B/Al composite indicated a better convergence to the isotropic case.

4.4. *A cross-crack under hydrostatic pressure*

Figure 6 shows the crack configuration and loading condition and presents the predicted results comparing them with those given by Cheung and Chen (1987) for the isotropic case. The stress intensity factors $K_I(A)/K_I^0$ and $K_I(B)/K_I^0$, ($K_I^0 = p^{\infty}\sqrt{\pi a}$) are in good

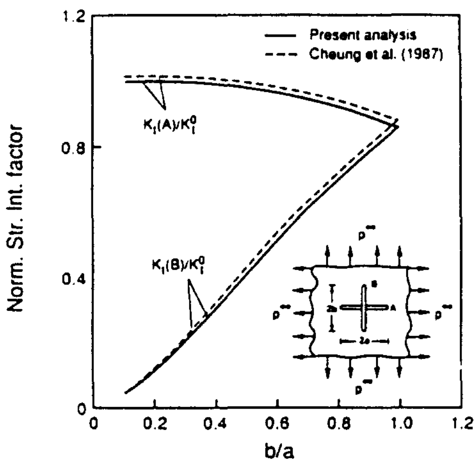

Fig. 6. The cross-shape configuration under hydrostatic pressure. The stress intensity factors $K_I(A)$ and $K_I(B)$ are normalized with respect to $K_I^0 = p^{\infty}\sqrt{\pi a}$. Isotropic results.

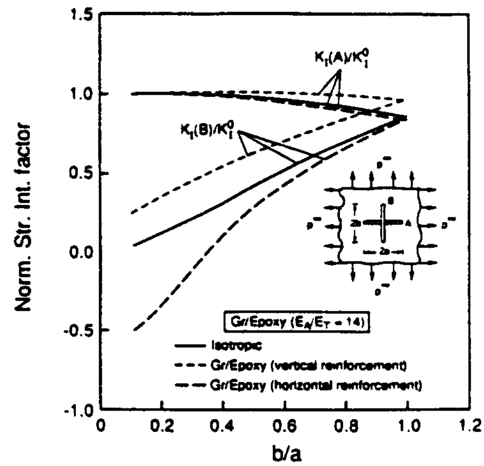

Fig. 7. The cross-shape configuration under hydrostatic pressure. The stress intensity factors $K_I(A)$ and $K_I(B)$ are normalized with respect to $K_I^0 = p^c\sqrt{\pi a}$. Isotropic and orthotropic (Gr/Epoxy) results.

agreement with those reported by Cheung and Chen (1987) over the entire range of b/a ratios.

Figures 7 and 8 present respectively the corresponding results for the Gr/Epoxy and $B/A\ell$ composites. In each case, the composite was considered to be reinforced axially or transversely. For the $B/A\ell$ composite, Fig. 7, the results are found to be almost identical to those of the isotropic case for $K_I(A)$ and almost symmetric with respect to the isotropic curve for $K_I(B)/K_I^0$. On the other hand, the results for the Gr/Epoxy composite, Fig. 8, are far apart from the isotropic curve for $K_I(B)$ but not so different for $K_I(A)$.

5. CONCLUSION

In summary, the problem of interacting cracks in orthotropic composites was considered. The method of superposition proposed herein indicated that the solution for complicated crack configuration can be obtained by the same simplicity as in the isotropic materials. In the $B/A\ell$ and Gr/Epoxy composites, it was observed that for cracks which are aligned with the fiber axis, the stress intensity factors and the E_A/E_T ratio have an inverse relationship.

The numerical results obtained herein are valuable for understanding the behavior of crack branching as well as stress shielding/amplification in some anisotropic materials.

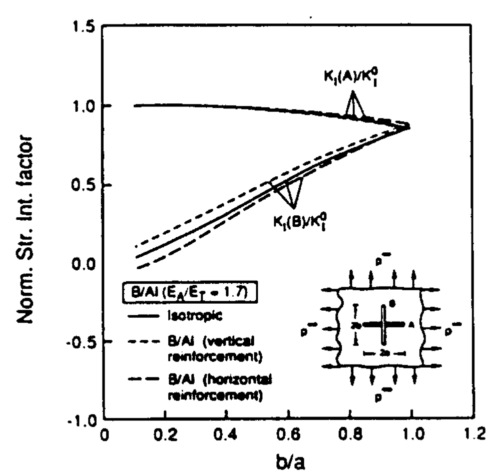

Fig. 8. The cross-shape configuration under hydrostatic pressure. The stress intensity factors $K_I(A)$ and $K_I(B)$ are normalized with respect to $K_I^0 = p^c\sqrt{\pi a}$. Isotropic and orthotropic ($B/A\ell$) results.

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APPENDIX: DERIVATIONS OF POWER-TYPE TRACTIONS

Consider a single crack of length $2l$ located in an infinte orthotropic medium. Let the crack be subjected to concentrated normal and shear loads P and Q as shown in Fig. A1.

A coordinate system (x, y) is located in the middle of the crack and the concentrated loads are applied at $(t, 0)$. Since the loads P and Q as appearing in that figure result in negative normal and shear stress respectively adjacent to the point of application, they will be assigned negative values, $P < 0$, $Q < 0$.

The problems of two-dimensional orthotropic elasticity can be reduced to finding two complex stress functions $\phi(z_1)$ and $\psi(z_2)$ where the complex variables z_j $(j = 1, 2)$ are given by

$$z_1 = x + s_1 y, \quad z_2 = x + s_2 y, \tag{A1}$$

with the parameters s_1 and s_2 being the roots of the characteristics equation [see Lekhnitskii (1963)]

$$a_{11}s^4 + 2a_{16}s^3 + (2a_{12} + a_{66})s^2 - 2a_{26}s + a_{22} = 0. \tag{A2}$$

The a_{ij} are the flexibility coefficients in generalized plane stress defined by:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{16} \\ a_{21} & a_{22} & a_{26} \\ a_{16} & a_{26} & a_{66} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}. \tag{A3}$$

For plane orthotropic problems, the coefficients of shear coupling a_{16} and a_{26} vanish and (A2) reduces to

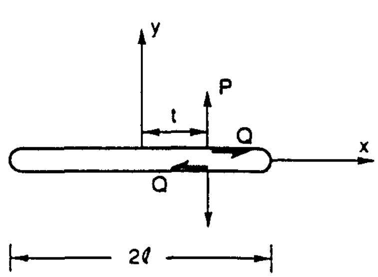

Fig. A1. A solitary crack loaded by concentrated shear and normal unit loads.

$$S^4+\left(\frac{2a_{12}+a_{66}}{a_{11}}\right)S^2+\frac{a_{22}}{a_{11}}=0. \tag{A4}$$

It has been shown by Lekhnitskii (1963) that the roots of (A4) are always complex or purely imaginary and will occur in conjugate pairs, i.e. $s_1, \bar{s}_1$ and $s_2, \bar{s}_2$. Without loss in generality, it is always possible to choose

$$s_1 = a_1+ib_1, \quad s_2 = a_2+ib_2, \tag{A5}$$

such that $b_1 > 0$, $b_2 > 0$ and $b_1 \neq b_2$.

The general expression for the stresses are then:

$$\begin{aligned}\sigma_{xx} &= 2\,\mathrm{Re}\,[s_1^2\phi(z_1)+s_2^2\psi(z_2)],\\ \sigma_{yy} &= 2\,\mathrm{Re}\,[\phi(z_1)+\psi(z_2)],\\ \sigma_{xy} &= -2\,\mathrm{Re}\,[s_1\phi(z_1)+s_2\psi(z_2)].\end{aligned} \tag{A6}$$

If the roots in (A4) are known for a reference system (x, y) then for any other reference (x', y') rotated at an angle θ from (x, y), the new roots are found from the following formula:

$$a'_{1,2}+ib'_{1,2}=\left(\frac{a_{1,2}\cos(2\theta)+1/2(a_{1,2}^2+b_{1,2}^2)\sin(2\theta)}{[\cos(\theta)+a_{1,2}\sin(\theta)]^2+[b_{1,2}\sin(\theta)]^2}\right)+i\left(\frac{b_{1,2}}{[\cos(\theta)+a_{1,2}\sin(\theta)]^2+[b_{1,2}\sin(\theta)]^2}\right). \tag{A7}$$

In particular, if the x-axis coincides with the fiber orientation, then

$$s_1 = ib_1 \quad \text{and} \quad s_2 = ib_2,$$

hence

$$s'_{1,2}=\left(\frac{1/2(b_{1,2}^2-1)\sin(2\theta)}{[\cos(\theta)]^2+[b_{1,2}\sin(\theta)]^2}\right)+i\left(\frac{b_{1,2}}{[\cos(\theta)]^2+[b_{1,2}\sin(\theta)]^2}\right). \tag{A8}$$

For the concentrated load on an isolated crack, the potential functions $\phi_n(z_1)$, $\psi_n(z_2)$ and $\phi_t(z_1)$, $\psi_t(z_2)$ for the P and Q loadings respectively, are given by Sih and Liebowitz (1968):

$$\begin{aligned}&[(s_2-s_1)/s_2]\phi_n(z_1) = -Pf(z_1), \qquad (s_2-s_1)\phi_t(z_1) = -Qf(z_1), \quad P<0,\\ &(s_1-s_2)\psi_t(z_2) = -Qf(z_2), \quad [(s_1-s_2)/s_1]\psi_n(z_2) = -Pf(z_2), \quad Q<0,\end{aligned} \tag{A9}$$

where

$$f(z) = [2\pi(z-t)]^{-1}[(\ell^2-t^2)/(z^2-\ell^2)]^{1/2}. \tag{A10}$$

As in the isotropic case, implementation of the crack interaction scheme to orthotropic media necessitates the use of power-type traction distributions (uniform, linear, quadratic, cubic and quartic). The corresponding stress potential functions can be derived by following a procedure of contour integration described by Muskhelishvili (1953).

In Mode I (opening mode) the stress potential functions are obtained from:

$$\left(\frac{s_2-s_1}{s_2}\right)\phi_\alpha(z_1)=\int_{-\ell}^{\ell}\frac{(t/\ell)^\alpha}{2\pi(z_1-t)}\left[\frac{\ell^2-t^2}{z_1^2-\ell^2}\right]^{1/2}\mathrm{d}t, \tag{A11a}$$

and similarly

$$\left(\frac{s_1-s_2}{s_1}\right)\psi_\alpha(z_2)=\int_{-\ell}^{\ell}\frac{(t/\ell)^\alpha}{2\pi(z_2-t)}\left[\frac{\ell^2-t^2}{z_2^2-\ell^2}\right]^{1/2}\mathrm{d}t, \tag{A11b}$$

where $\alpha = (0, 1, 2, 3, 4)$ indicating, respectively, uniform, linear, quadratic, cubic and quartic terms.

The corresponding stress potential functions are:

Uniform load:

$$\begin{aligned}\left(\frac{s_2-s_1}{s_2}\right)\phi_\alpha(z_1) &= [2(z_1^2-\ell^2)^{1/2}]^{-1}[z_1-(z_1^2-\ell^2)^{1/2}],\\ \left(\frac{s_1-s_2}{s_1}\right)\psi_\alpha(z_2) &= [2(z_2^2-\ell^2)^{1/2}]^{-1}[z_2-(z_2^2-\ell^2)^{1/2}].\end{aligned} \tag{A12a}$$

Linear load:

$$\left(\frac{s_2-s_1}{s_2}\right)\phi_n(z_1) = [2\ell(z_1^2-\ell^2)^{1/2}]^{-1}\left[z_1^2-z_1(z_1^2-\ell^2)^{1/2} = \frac{\ell^2}{2}\right],$$

$$\left(\frac{s_1-s_2}{s_1}\right)\psi_n(z_2) = [2\ell(z_2^2-\ell^2)^{1/2}]^{-1}\left[z_2^2-z_2(z_2^2-\ell^2)^{1/2} - \frac{\ell^2}{2}\right]. \tag{A12b}$$

Quadratic load :

$$\left(\frac{s_2-s_1}{s_2}\right)\phi_n(z_1) = [2\ell^2(z_1^2-\ell^2)^{1/2}]^{-1}[z_1^3-z_1^2(z_1^2-\ell^2)^{1/2}-\tfrac{1}{2}z_1\ell^2],$$

$$\left(\frac{s_1-s_2}{s_1}\right)\psi_n(z_2) = [2\ell^2(z_2^2-\ell^2)^{1/2}]^{-1}[z_2^3-z_2^2(z_2^2-\ell^2)^{1/2}-\tfrac{1}{2}z_2\ell^2]. \tag{A12c}$$

Cubic load :

$$\left(\frac{s_2-s_1}{s_2}\right)\phi_n(z_1) = [2\ell^2(z_1^2-\ell^2)^{1/2}]^{-1}\left[z_1^4-z_1^3(z_1^2-\ell^2)^{1/2}-z_1^2\frac{\ell^2}{2}-\tfrac{1}{8}\ell^4\right],$$

$$\left(\frac{s_1-s_2}{s_1}\right)\psi_n(z_2) = [2\ell^3(z_2^2-\ell^2)^{1/2}]^{-1}\left[z_2^4-z_2^3(z_2^2-\ell^2)^{1/2}-z_2^2\frac{\ell^2}{2}-\tfrac{1}{8}\ell^4\right]. \tag{A12d}$$

Quartic load :

$$\left(\frac{s_2-s_1}{s_2}\right)\phi_n(z_1) = [2\ell^4(z_1^2-\ell^2)^{1/2}]^{-1}\left[z_1^5-z_1^4(z_1^2-\ell^2)^{1/2}-\frac{z_1^3\ell^2}{2}-z_1\frac{\ell^4}{8}\right],$$

$$\left(\frac{s_1-s_2}{s_1}\right)\psi_n(z_2) = [2\ell^4(z_2^2-\ell^2)^{1/2}]^{-1}\left[z_2^5-z_2^4(z_2^2-\ell^2)^{1/2}-\frac{z_2^3\ell^4}{2}-z_2\frac{\ell^2}{8}\right]. \tag{A12e}$$

In mode II (shearing mode), all loading-type stress potentials can be obtained from those of mode I using the following relationship :

$$\frac{1}{s_2}\phi_n(z_1) = \phi_t(z_1), \quad \frac{1}{s_1}\psi_n(z_2) = \psi_t(z_2). \tag{A13}$$

Substitution of (A12)–(A13) into (A6) provides the influence functions which constitute the basic ingredient of the method. Although the influence functions are evaluated here in terms of complex potential functions, they can actually be expressed in elementary functions [see Zarzour (1990)].